Bsp. 35 Linienschwerpunkt: dL nötig für Formel

$$dL = \sqrt{dx^2 + dy^2} \quad \text{diff. Längenelement}, \quad \text{mit} \quad dy = \frac{d}{dx}(0.5x^2)\,dx = x\,dx$$

SP: $$\bar{x} = \frac{\int x\,dL}{\int dL} = \frac{\int x\sqrt{dx^2+dy^2}}{\int \sqrt{dx^2+dy^2}} = \frac{\int_0^1 x\sqrt{dx^2+x^2dx^2}}{\int_0^1 \sqrt{dx^2+x^2dx^2}}$$

$$\int_0^1 dL = \int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1 \sqrt{1+\left(\frac{x\,dx}{dx}\right)^2}\,dx = \int_0^1 \sqrt{1+x^2}\,dx \underset{\substack{(\text{siehe ZI})\\(\alpha)}}{=} \frac{1}{2}\left[x\sqrt{1+x^2} + \text{arsinh}(x)\right]_0^1 = \underline{1.148\,m}$$

Folie einfügen & dort wischen

$$\int_0^1 x\,dL = \int_0^1 x\sqrt{1+x^2}\,dx = \left|\begin{matrix} u = 1+x^2 \\ du = 2x\,dx \rightarrow dx = \frac{du}{2x}\end{matrix}\right| = \int_1^2 x\sqrt{u}\,\frac{du}{2\cancel{x}} = \frac{1}{2}\int_1^2 u^{1/2}\,du = \frac{1}{\cancel{2}}\,\frac{\cancel{2}u^{3/2}}{3}\Big|_1^2 = \underline{0.609\,m^2}$$

$$\boxed{\bar{x} = \frac{0.609\,m^2}{1.148\,m} = 0.531\,m}$$

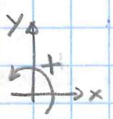
$$\bar{y} = \frac{\int y\,dL}{\int dL}, \quad \int_0^1 dL = 1.148\,m \quad \text{siehe oben.}$$

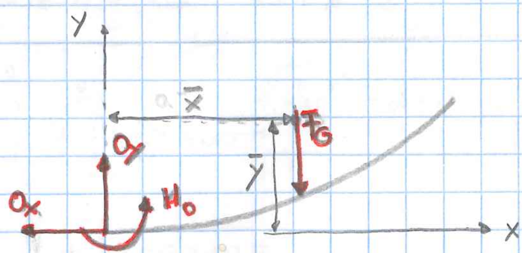
$$\int y\,dL = \int_0^1 \underbrace{0.5x^2}_{y(x)}\underbrace{\sqrt{1+x^2}\,dx}_{dL(x,dx)} = 0.5\int_0^1 x^2\sqrt{1+x^2}\,dx = 0.5\cdot\left[\frac{1}{4}x\sqrt{(1+x^2)^3} - \frac{1}{8}\left(x\sqrt{1+x^2} + \ln\left(x+\sqrt{1+x^2}\right)\right)\right]_0^1$$

$$= 0.2101\,m^2$$

$$\boxed{\bar{y} = \frac{0.2101\,m^2}{1.148\,m} = 0.183\,m}$$

Reaktionskräfte in Einspannung:





$$\sum F_{x_i} = 0: \quad \underline{O_x = 0}$$

$$\sum F_{y_i} = 0: \quad O_y - F_G = 0, \quad F_G = q_G \cdot L = 5\,N/m \cdot 1.148\,m = 5.74\,N$$

$$\underline{O_y} = F_G = \underline{5.74\,N}$$

$$\sum M_z^{(0)} = 0: \quad M_0 - F_G\cdot\bar{x} = 0$$

$$\underline{M_0} = F_G\cdot\bar{x} = 5.74\,N\cdot 0.531\,m = 3.048\,Nm$$

(d) $\int \sqrt{1+x^2}\,dx = \left| \begin{matrix} x = \sinh(u) \\ dx = \cosh(u)du \end{matrix} \right| = \int \sqrt{1+\sinh^2(u)}\,\cosh(u)\,du = \left| \begin{matrix} \cosh^2(u) - \sinh^2(u) = 1 \\ \cosh^2(u) = 1 + \sinh^2(u) \end{matrix} \right|$

$= \int \sqrt{\cosh^2(u)}\,\cosh(u)\,du = \int \cosh^2(u)\,du$

Löse also $\int \cosh^2(u)\,du = \frac{1}{2}\int (1 + \cosh(2u))\,du = \int \frac{1}{2}\,du + \int \frac{1}{2}\cosh(2u)\,du = \frac{1}{2}\left[u + \sinh(2u)\frac{1}{2}\right] + c$

$= \frac{u}{2} + \sinh(u)\cosh(u)\frac{\cancel{2}}{\cancel{4}_2} + c = \frac{1}{2}\left(u + \sinh(u)\cosh(u)\right) + c$

$\sinh(2u) = 2\sinh(u)\cosh(u)$

$= \left| \begin{matrix} u = \operatorname{arsinh}(x) \\ \cosh(u) = \sqrt{1+\sinh^2(u)} \\ = \sqrt{1+x^2} \end{matrix} \right| = \underline{\frac{1}{2}\left[\operatorname{arsinh}(x) + x\sqrt{1+x^2}\right]}$

wobei benutzt wurde: $\cosh(2u) = \cosh(u)\cosh(u) + \sinh(u)\sinh(u) = \cosh^2(u) + \sinh^2(u)$

$= \cosh^2(u) + \underbrace{\cosh^2(u) - 1}_{\sinh^2(u)} = 2\cosh^2(u) - 1$

$\longrightarrow \underline{\cosh^2(u) = \frac{1}{2}\left(1 + \cosh(2u)\right)}$

Bsp.: 36

diff. Flächenelement $\quad dA = y\,dx = 2K\left(x - \frac{x^2}{2a}\right)dx$

$$\bar{x} = \frac{\int_A x\,dA}{\int_A dA} \quad , \quad \int_A dA = \int_0^a 2K\left(x - \frac{x^2}{2a}\right)dx = 2K\left(\int_0^a x\,dx - \int_0^a \frac{x^2}{2a}\,dx\right) =$$

$$= \cancel{2}K\frac{x^2}{\cancel{2}}\Big|_0^a - \frac{\cancel{2}K}{\cancel{2}a}\frac{x^3}{3}\Big|_0^a = K\left(a^2 - \frac{a^{\cancel{3}2}}{3\cancel{a}}\right) = \underline{\frac{2Ka^2}{3}}$$

$$\int_A x\,dA = 2K\left(\int_0^a x^2\,dx - \frac{1}{2a}\int_0^a x^3\,dx\right) = 2K\frac{x^3}{3}\Big|_0^a - \frac{K}{a}\frac{x^4}{4}\Big|_0^a =$$

$$= \frac{2Ka^3}{3} - \frac{Ka^3}{4} = \frac{8Ka^3}{12} - \frac{3Ka^3}{12} = \underline{\frac{5Ka^3}{12}}$$

$$\underline{\bar{x}} = \frac{5Ka^{\cancel{3}}\cdot 3}{12\cdot 2\cancel{Ka^2}} = \frac{15a}{24} = \underline{\frac{5a}{8}}$$

Bsp. 37:

$dA = \frac{1}{2}r^2\,d\theta \qquad$ vgl. Fläche eines Kreissektors: $A_{KS} = \frac{1}{2}r^2\theta$ (Vollkreis $\frac{1}{2}r^2 2\pi = r^2\pi$)

Öffnungswinkel des Sektors

Rechentrick: Nur Fläche von halbem Kardioid berechnen (obere Hälfte) & diese doppelt zählen.

$$2\int_A dA = \cancel{2}\int_0^\pi \frac{1}{\cancel{2}}\underbrace{a^2(1-\cos\theta)^2}_{r^2}d\theta = \int_0^\pi a^2(1 - 2\cos\theta + \cos^2\theta)\,d\theta = a^2\left(\int_0^\pi d\theta - \int_0^\pi 2\cos\theta\,d\theta + \int_0^\pi \cos^2\theta\,d\theta\right)$$

$$= a^2\theta\Big|_0^\pi \underbrace{-2a^2\sin(\theta)\Big|_0^\pi}_{\text{beide}=0} + a^2\left(\frac{\theta}{2} + \frac{\cos\theta\sin\theta}{2}\right)\Big|_0^\pi = a^2\pi + \frac{\pi}{2}a^2 = \underline{\frac{3\pi}{2}a^2}$$

Folie einfügen:

NB: $\int_a^b \cos^2 x\,dx = \int_a^b \cos x\cos x\,dx = \int_a^b \underbrace{\cos x}_{f}\,\underbrace{(\sin x)'}_{g'}\,dx = \underbrace{\cos x}_{f}\,\underbrace{\sin x}_{g}\Big|_a^b - \int_a^b \underbrace{(\cos x)'}_{f'}\,\underbrace{\sin x}_{g}\,dx$

PI: $\int f g'\,dx = fg - \int f' g\,dx$

$$= \cos x\sin x\Big|_a^b + \int_a^b \sin^2 x\,dx = \cos x\sin x\Big|_a^b + \int_a^b (1 - \cos^2 x)\,dx$$

$$= \cos x\sin x\Big|_a^b + \int_a^b dx - \int_a^b \cos^2 x\,dx$$

$$\to 2\int_a^b \cos^2 x\,dx = \cos x\sin x\Big|_a^b + x\Big|_a^b \quad \to \quad \underline{\int_a^b \cos^2 x\,dx = \frac{1}{2}(x + \cos x\sin x)\Big|_a^b}$$

⚠ Wichtiger Trick für derartige Integrale!
Funktioniert analog für $\int \sin^2 x\,dx$!

$$r^2\cos\theta\, dr\, d\theta = a^2(1-\cos\theta)^2\cos\theta\, dr\, d\theta$$

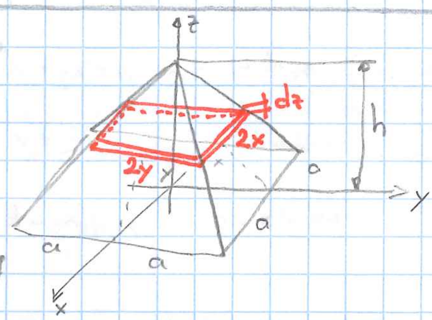
$$\int_A \bar{r}\, dA = 2\int_0^{\pi} \underbrace{\frac{2}{3} r\cos\theta}_{\bar{r}(\theta)}\; \underbrace{\frac{1}{2}a^2(1-\cos\theta)^2 d\theta}_{dA} = \frac{2}{3}a^3\int_0^{\pi} (1-\cos\theta)^3\cos\theta\, d\theta$$

Tabelle od. CAS

$$= \frac{16}{3}a^3\left[-\frac{15\pi}{64} + \underbrace{\frac{13\sin\pi}{32}}_{=0} - \frac{1}{8}\underbrace{\sin(2\pi)}_{=0} + \frac{1}{32}\underbrace{\sin(3\pi)}_{=0} - \frac{1}{256}\underbrace{\sin(4\pi)}_{=0}\right] = -\frac{240\pi}{192}a^3 = -3.926a^3$$

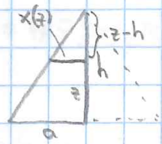
$$\bar{r} = \frac{-3.926\,a^3}{\frac{3\pi}{2}a^2} = \underline{-0.833a}$$

Bsp. 38:



$dV = 2x\,2y\,dz = 4xy\,dz$

Um $x(z), y(z)$ zu erhalten wird Ähnlichkeitsbetrachtung benötigt:



Es gilt: $\frac{h}{a} = \frac{h-z}{x(z)}$ also $x(z) = \frac{a}{h}(h-z)$

und analog für $y(z)$, sodass $y(z) = x(z)$ (weil ja Seitenlänge a gleich)

$$dV = 4\frac{a^2}{h^2}(h-z)^2 dz$$

$$\int_V dV = \int_0^h 4\frac{a^2}{h^2}(h-z)^2 dz = \frac{4a^2}{h^2}\left(\int_0^h h^2 dz - \int_0^h 2hz\, dz + \int_0^h z^2 dz\right) = \frac{4a^2}{h^2}\left(h^2 z\Big|_0^h - 2h\frac{z^2}{2}\Big|_0^h + \frac{z^3}{3}\Big|_0^h\right) =$$

$$= \frac{4a^2}{h^2}\left(\cancel{h^3} - \cancel{h^3} + \frac{h^3}{3}\right) = \underline{\frac{4a^2}{3}h}$$

(Vgl. Volumsformel f. Pyr.: $V = \frac{1}{3}G\cdot h = \frac{1}{3}(2a)^2 h = \underline{\frac{4a^2}{3}h}$)

$$\int_V z\, dV = \int_0^h 4\frac{a^2}{h^2}(h-z)^2 z\, dz = \frac{4a^2}{h^2}\left(\int_0^h h^2 z\, dz - \int_0^h 2hz^2 dz + \int_0^h z^3 dz\right)$$

$$= \frac{4a^2}{h^2}\left(h^2\frac{z^2}{2}\Big|_0^h - 2h\frac{z^3}{3}\Big|_0^h + \frac{z^4}{4}\Big|_0^h\right) = \frac{4a^2}{\cancel{h^2}}\left(\frac{h^{\cancel{4}\,2}}{2} - \frac{2h^{\cancel{4}\,2}}{3} + \frac{h^{\cancel{4}\,2}}{4}\right) = 4a^2\left(\frac{6h^2}{12} - \frac{8h^2}{12} + \frac{3h^2}{12}\right)$$

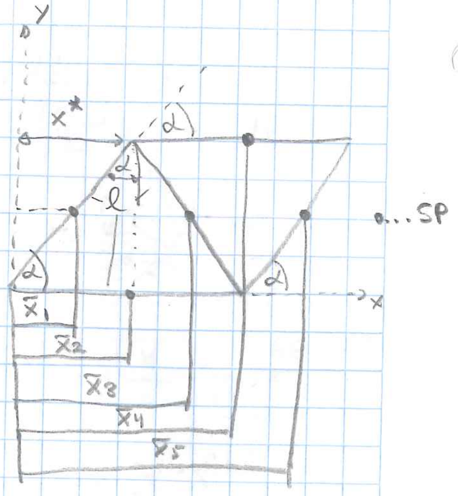
$$= \frac{4a^2h^2}{12} = \underline{\frac{a^2h^2}{3}}$$

$$\bar{z} = \frac{\int_V z\, dV}{\int_V dV} = \frac{\cancel{a^2} h^{\cancel{2}}\, \cancel{3}}{\cancel{3}\, 4\cancel{a^2}\cancel{h}} = \frac{h}{4} \longrightarrow \underline{\underline{\bar{z} = \frac{h}{4}}} \quad \text{q.e.d.}$$

Bsp. 39:

Strategie: d muss im Schwerpunktsabstand $\bar{x}$ liegen.

$$d \overset{!}{=} \bar{x} = \frac{\sum \bar{x} M}{\sum M}$$



wobei $\sum M = 5 \cdot l \cdot m = 5 \cdot 4m \cdot 7\,Kgm^{-1} = \underline{140\,Kg}$

$$\sum \bar{x} M = l \cdot m \cdot (\bar{x}_1 + \bar{x}_2 + \bar{x}_3 + \bar{x}_4 + \bar{x}_5)$$

$\bar{x}_1 = \frac{l}{2}\cos\alpha = 1m$, $\bar{x}_3 = x^* + \frac{l}{2}\cos\alpha = 3m$

$\bar{x}_2 = \frac{l}{2} = 2m$, $\bar{x}_4 = x^* + \frac{l}{2} = 4m$

$x^* = l \cdot \cos\alpha = 2m$, $\bar{x}_5 = l + \frac{l}{2}\cos\alpha = 5m$

$$\sum \bar{x} M = 4m \cdot 7\,Kgm^{-1} \cdot (1m + 2m + 3m + 4m + 5m) = \underline{420\,Kgm}$$

$$\underline{d} = \frac{\sum \bar{x} M}{\sum M} = \frac{420\,Kgm}{140\,Kg} = \underline{3m}$$

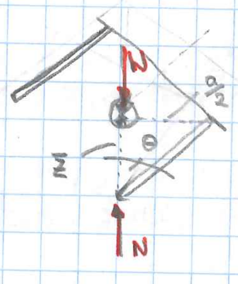
Bsp. 40:

$$\bar{z} = \frac{\sum \bar{z} N}{\sum N} \quad \text{mit} \quad \sum N = \overset{\text{Platte}}{G_1} + 4 \cdot \overset{\text{Beine}}{G_2} = 75N + 4 \cdot 10N = \underline{115N}$$

$$\sum \bar{z} N = G_1 \bar{z}_1 + 4 \cdot G_2 \cdot \bar{z}_2 = 75N \cdot 1m + 4 \cdot 10N \cdot 0.5m$$

$$= \underline{95Nm}$$

$$\underline{\bar{z}} = \frac{95Nm}{115N} = \underline{0.826m}$$



Kippen: $\tan\theta = \frac{a/2}{\bar{z}}$

$$\underline{\theta} = \tan^{-1}\left(\frac{a/2}{\bar{z}}\right) = \underline{42.24°}$$

Bsp. 41:

diff. Flächenelement: $dA = x\,dy = \left(\overbrace{\frac{a}{2}}^{\text{Rechteck}} + \overbrace{\frac{y^2}{a}}^{\text{Kurve}}\right) dy$

$$A = \int_A dA = \int_0^a \left(\frac{a}{2} + \frac{y^2}{a}\right) dy = \int_0^a \frac{a}{2}\,dy + \int_0^a \frac{y^2}{a}\,dy = \frac{a}{2} y \Big|_0^a + \frac{1}{a}\frac{y^3}{3}\Big|_0^a = \frac{a^2}{2} + \frac{a^2}{3}$$

$$= \underline{\frac{5}{6} a^2} = \underline{3.\dot{3}\,m^2}$$

$$\int_A y\,dA = \int_0^a \frac{ay}{2}\,dy + \int_0^a \frac{y^3}{a}\,dy = \frac{ay^2}{4}\Big|_0^a + \frac{y^4}{4a}\Big|_0^a = \frac{a^3 + a^3}{4} = \frac{a^3}{2} = \underline{4m^3}$$

$$\bar{y} = \frac{\int_A y\,dA}{\int_A dA} = \frac{4\,m^3}{3.\dot{3}\,m^2} = \underline{1.2\,m}$$

Volumen über Pappus-Guldin'sche Regel:

Volumen = erzeugende Fläche x Kreisbogen des Flächenschwerpunktes bei Drehung

Beweis: siehe z.B. Hibbeler 1, S. 511 (ist aber unmittelbar einsichtig!)

d.h. $V = \theta F A$ mit θ ... Rotationswinkel im Bogenmaß, $\theta \leq 2\pi$
F ... senkrechter Abstand Rotationsachse zu Flächenschwp.
A ... erzeugende Fläche

Für Bsp gilt daher: $\theta = 2\pi$, $F = \bar{y}$, $A = \int_A dA$

$$\underline{V} = \theta \bar{y} A = 2\pi \cdot 1.2\,m \cdot 3.\dot{3}\,m^2 = \underline{25.1\,m^3}$$

Bsp. 42: Schwerpunkt als Summe von Einzelschwerpunkten: $\bar{y} = \frac{\sum \bar{y} L}{\sum L} = \frac{[m^2]}{[m]} = [m]$

$$\sum L = 4 \cdot a_1 + 2 \cdot a_2 + 2 \cdot b_1 + 2 \cdot b_2 = \underline{20\,cm}$$

$$\sum \bar{y} L = 3\bar{y}_{a_1} \cdot a_1 + 2\bar{y}_{b_2} \cdot b_2 + 2\bar{y}_{b_1} \cdot b_1 + 2\bar{y}_{a_2} \cdot a_2 + \bar{y}_{a^*} \cdot a_1$$

$$= 0 \cdot 3a_1 + \frac{b_2}{\cancel{2}} \cdot \cancel{2} b_2 + \frac{b_1}{\cancel{2}} \cdot \cancel{2} b_1 + b_1 \cdot 2a_2 + b_2 \cdot a_1$$

$$= \underline{12\,cm^2}$$

$$\underline{\bar{y}} = \frac{\sum \bar{y} L}{\sum L} = \frac{12\,cm^2}{20\,cm} = \underline{0.6\,cm}$$

nur eine Seite (links) & alles doppelt rechnen:

i	L_i	$\bar{y}_i$	$\bar{y}_i L_i$	$\bar{y}_i L_i$-Wert $[cm^2]$
1	a_2	b_1	$a_2 \cdot b_1$	2
2	b_1	$b_1/2$	$b_1^2/2$	2
3	$1.5a_1$	0	0	0
4	b_2	$b_2/2$	$b_2^2/2$	0.5
5	$a_1/2$	b_2	$a_1/2\, b_2$	1.5
$2 \cdot \sum$	20	/	/	12

$$\rightarrow \underline{\bar{y}} = \frac{\sum_i \bar{y}_i L_i}{\sum_i L_i} = \underline{\underline{0.6\,cm}}$$